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Soliton Perturbation Theory for the Compound KdV Equation

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The soliton perturbation theory is used to study the solitons that are governed by the compound Korteweg de-Vries equation in presence of perturbation terms. The adiabatic parameter dynamics of the solitons in presence of the perturbation terms are obtained.

KEY WORDS: soliton perturbation theory; adiabatic parameter dynamics; KdV equation.

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1. INTRODUCTION

The compound Korteweg de-Vries (cKdV) equation that is going to be studied in this paper arises is in the study of nonlinear waves in fluids. In particular, it shows up in the context of internal gravity waves in a density-stratified ocean. A very common tool for describing these processes is the Korteweg-de Vries (KdV) equation and its modifications that are valid for small nonlinearity. But, in some cases, really strong nonlinear waves are observed. An example of this is the Coastal Ocean Probe Experiment (COPE) that was carried out in September 1995 in the Oergon Bay. The data of this experiment clearly indicated the presence of extremely strong trains of internal waves propagating to the shore. They had a form of isothermal depression consisting of solitary pulses with amplitudes of thermocline displacenments typically of order 10–20 m, and sometimes reaching 30 m. The character of these trains (internal bores) is rather typical of the tidally generated waves on shelves, but 30 m high solitons are usually encountered in deep ocean rather than on the shelf. More importantly, it is a fact that they propagate into the background of a very shallow, 5–7 m depth density jump (pynocline). This

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means that the soliton amplitude exceed the background by upto 5–6 times, and testifies for an extremely strong nonlinearity for the process, namely a high Mach number, using a terminology from gas dynamics. Thus, the problem of creating an adequate theoretical model of these processes is evidently necessary.

The dimensionless form of the cKdV equation that is going to be studied in this paper is given by

$$q_t + (aq^p + bq^{2p})\, q_x + cq_{xxx} = 0 \tag{1}$$

Here, in Equation (1), a, b, c and p are constants and $p > 1$. For (1), kink as well as bell-profile solitary wave solutions exist for $p = 1$. In fact, this equation is also studied in quantum field theory, plasma physics and solid state physics. For example, the kink soliton can be can be used to calculate energy and momentum flow and topological charge in the quantum field.

2. MATHEMATICAL PROPERTIES

Equation (1) is not integrable by the method of Inverse Scattering Transform (IST) which is a common technique that is used to integrate many nonlinear evolution equations. One can easily see that (1) will fail the Painleve test of integrability. However, Equation (1) supports solitons of the form

$$q = \left[\frac{A}{K + 2\cosh^2 B(x - \bar{x})}\right]^{\frac{1}{p}} \tag{2}$$

where

$$A = v(p+1)(p+2)\sqrt{\frac{2p+1}{a^2(2p+1) + bv(p+1)(p+2)^2}} \tag{3}$$

$$K = -1 + a\sqrt{\frac{2p+1}{a^2(2p+1) + bv(p+1)(p+2)^2}} \tag{4}$$

and

$$B = \frac{p}{2}\sqrt{\frac{v}{c}} \tag{5}$$

In (2), A represents the amplitude of the soliton, B represents the inverse width of the soliton, while v is the velocity of the soliton and $\bar{x}$ is the center of the soliton, so that

$$v = \frac{d\bar{x}}{dt} \tag{6}$$

This solution given by (2) was first seen in 2002 (Li and Liu, 2002; Zhang *et al.*, 2002). The method that was used to obtain these solution was the travelling wave ansatz. Equation (1) has at least three integrals of motion that are known as linear momentum (M), energy (E) and the Hamiltonian (H) (Zhidkov, 2001). These are respectively given by

$$M = \int_{-\infty}^{\infty} q\,dx = \frac{4}{B}\left(\frac{A}{2}\right)^{\frac{1}{p}} F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p} + \frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{1}{p}\right) \tag{7}$$

and

$$E = \int_{-\infty}^{\infty} q^2 dx = \frac{4}{B}\left(\frac{A}{2}\right)^{\frac{2}{p}} F\left(\frac{2}{p}, \frac{2}{p}, \frac{2}{p} + \frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{2}{p}\right) \tag{8}$$

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \left\{\frac{1}{2}q_x^2 - \frac{a}{(p+1)(p+2)}q^{p+2} - \frac{b}{(2p+1)(2p+2)}q^{2p+2}\right\} dx \\ &= \frac{16B}{p^2}\frac{A^{\frac{2}{p}}}{2^{\frac{1}{p}}} F\left(\frac{1}{p}+1, \frac{1}{p}-1, \frac{1}{p}-\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{3}{2}, \frac{1}{p}-1\right) \\ &\quad - \frac{2a}{B(p+1)(p+2)}\frac{A^{\frac{p+2}{p}}}{2^{\frac{2}{p}}} F\left(\frac{p+2}{p}, \frac{p+2}{p}, \frac{p+2}{p}+\frac{1}{2}; -\frac{K}{2}\right) \\ &\quad \times B\left(\frac{1}{2}, \frac{p+2}{p}\right) \\ &\quad - \frac{b}{B(2p+1)(2p+2)}\frac{A^{\frac{2p+2}{p}}}{2^{\frac{2}{p}}} F\left(\frac{2p+2}{p}, \frac{2p+2}{p}, \frac{2p+2}{p}+\frac{1}{2}; -\frac{K}{2}\right) \\ &\quad \times B\left(\frac{1}{2}, \frac{2p+2}{p}\right) \end{aligned} \tag{9}$$

These conserved quantities are calculated by using the 1-soliton solution given by (2). Also, in (7)–(9), $F(\xi, \eta, \zeta; z)$ is the Gauss' hyper-geometric function defined as

$$F(\xi, \eta, \zeta; z) = \frac{\Gamma(\zeta)}{\Gamma(\xi)\Gamma(\eta)} \sum_{n=0}^{\infty} \frac{\Gamma(\xi+n)\Gamma(\eta+n)}{\Gamma(\zeta+n)} \frac{z^n}{n!} \tag{10}$$

and $B(u, v)$ is the beta function that is defined as

$$B(u, v) = \int_0^1 x^{u-1}(1-x)^{v-1} dx \tag{11}$$

The center of the soliton $\bar{x}$ is given by the definition

$$\bar{x} = \frac{\int_{-\infty}^{\infty} xq\,dx}{\int_{-\infty}^{\infty} q\,dx} = \frac{\int_{-\infty}^{\infty} xq\,dx}{M} \tag{12}$$

where M is defined in (10). Thus, the velocity of the soliton is given by

$$v = \frac{d\bar{x}}{dt} = \frac{\int_{-\infty}^{\infty} x q_t dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} x q_t dx}{M} \tag{13}$$

On using (1) and (7), the velocity of the soliton reduces to

$$\begin{aligned} v = & \frac{aA}{2(p+1)} \frac{F\left(\frac{p+1}{p}, \frac{p+1}{p}; \frac{p+1}{p} + \frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{p+1}{p}\right)}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p} + \frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{1}{p}\right)} \\ & + \frac{bA^2}{4(2p+1)} \frac{F\left(\frac{2p+1}{p}, \frac{2p+1}{p}; \frac{2p+1}{p} + \frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{2p+1}{p}\right)}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p} + \frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{1}{p}\right)} \end{aligned} \tag{14}$$

3. PERTURBATION TERMS

The perturbed cKdV equation that is going to be studied in this paper is given by

$$q_t + \left(aq^p + bq^{2p}\right) q_x + cq_{xxx} = \epsilon R \tag{15}$$

where, in (15), ϵ is the perturbation parameter and $0 < \epsilon \ll 1$, while R gives the perturbation terms. In this paper,

$$R = \alpha q + \beta q_{xx} + \gamma q^m q_x + \delta q_x^3 \tag{16}$$

In R, dissipation gives rise to the first two terms and so α and β are small dissipative coefficients (Kichenassamy, 1997; Kivshar and Malomed, 1989; Marchant and Smyth, 1996; Osborne, 1997; Ostrovsky and Stepanyants, 1989; Wazwaz, 2003). Also, γ represents the coefficient of higher order nonlinear dispersive term (Kichenassamy, 1997; Kivshar and Malomed, 1989; Marchant and Smyth, 1996; Osborne, 1997; Ostrovsky and Stepanyants, 1989; Wazwaz, 2003) and m is a positive integer with $1 \leq m \leq 4$ (Kivshar and Malomed, 1989).

In presence of these perturbation terms, the momentum, energy and Hamiltonian do not stay conserved. Instead they undergo adiabatic changes that lead to the adiabatic deformation of the soliton amplitude and a slow change in the velocity (Kodama and Ablowitz, 1981). Using (8), the law of adiabatic deformation of the soliton amplitude is given by Mann (1997)

$$\frac{dA}{dt} = \frac{\epsilon p B A^{1-\frac{2}{p}}}{2^{2-\frac{2}{p}}} \frac{1}{F\left(\frac{2}{p}, \frac{2}{p}, \frac{2}{p} + \frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{2}{p}\right)} \int_{-\infty}^{\infty} qR dx \tag{17}$$

Again, using the result for the velocity of the soliton given by (14) for the perturbed cKdV equation (16), one gets (Mann, 1997)

$$\begin{aligned} v = {} & \frac{aA}{2(p+1)} \frac{F\left(\frac{p+1}{p}, \frac{p+1}{p}; \frac{p+1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{p+1}{p}\right)}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{1}{p}\right)} \\ & + \frac{bA^2}{4(2p+1)} \frac{F\left(\frac{2p+1}{p}, \frac{2p+1}{p}; \frac{2p+1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{2p+1}{p}\right)}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{1}{p}\right)} \\ & + \frac{\epsilon}{M} \int_{-\infty}^{\infty} xR\,dx \end{aligned} \tag{18}$$

where M is given by (7). In presence of these perturbation terms given by (16), the adiabatic variation of the soliton amplitude is given by

$$\frac{dA}{dt} = \epsilon \left[\alpha p A - \frac{4\beta A B^2}{p} \frac{F\left(\frac{2}{p}+2, \frac{2}{p}, \frac{2}{p}+\frac{3}{2}; -\frac{K}{2}\right) B\left(\frac{3}{2}, \frac{2}{p}\right)}{F\left(\frac{2}{p}, \frac{2}{p}; \frac{2}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{2}{p}\right)} \right] \tag{19}$$

Finally, the velocity of the perturbed soliton for the cKdV equation, in presence of the perturbation terms in (16), is given by

$$\begin{aligned} v = {} & \frac{aA}{2(p+1)} \frac{F\left(\frac{p+1}{p}, \frac{p+1}{p}; \frac{p+1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{p+1}{p}\right)}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{1}{p}\right)} \\ & + \frac{bA^2}{4(2p+1)} \frac{F\left(\frac{2p+1}{p}, \frac{2p+1}{p}; \frac{2p+1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{2p+1}{p}\right)}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{1}{p}\right)} \\ & - \epsilon \left[\frac{\gamma}{m+1} \left(\frac{A}{p}\right)^{\frac{m}{p}} \frac{F\left(\frac{m+1}{p}, \frac{m+1}{p}, \frac{m+1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{2}{p}\right)}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{1}{p}\right)} \right. \\ & + \frac{16\delta B^2 A^{\frac{2}{p}} 2^{\frac{1}{p}}}{p^3} \frac{1}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{1}{2}; -\frac{K}{2}\right) B\left(\frac{1}{2}, \frac{1}{p}\right)} \\ & \left. \times \int_{-\infty}^{\infty} \frac{s \sinh^3 s \cosh^3 s}{\left(K + 2\cosh^2 s\right)^{\frac{3(p+1)}{p}}} \right] \end{aligned} \tag{20}$$

4. CONCLUSIONS

In this paper, soliton perturbation theory is used to study the peturbed cKdV equation. This theory is used to establish the adiabatic parameter dynamics of the soliton amplitude. Also, it is shown that the velocity undergoes a slow change due to these perturbation terms.

In future, it is possible to extend these perturbation terms to include other perturbation terms that include the non-local ones too. In addition, one can extend this study to consider other nonlinear evolution equations that include the Sawada-Kotera equation and its higher order generalizations (Feng and Kawahara, 2000a,b; Feng, 2002, 2003; Kaya, 2004; Parkes and Duffy, 1996; Wazwaz, 2006a,b). One can also take into account the generalized Fisher's equation as well as two-dimensional Boussinesq's equation (Johnson, 1996) and so forth.

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